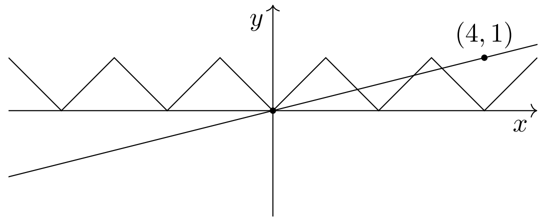
4701. The graphs of $y = \text{f}(x)$ and $y = \frac{1}{4}x$ are



The relevant points are the origin, where the graphs intersect, and $(4, 1)$, where they don't. Clearly, since the gradient of $y = |x|$ is ± 1, the graphs intersect four times in the domain $[0, 4]$. Outside that domain, $\frac{1}{4}x$ is either negative or greater than 1, so cannot intersect again. Hence, $\text{f}(x) - \frac{1}{4}x = 0$ has precisely four roots.

4702. Scale the triangles to have side length 1.

The six right-angled triangles have interior angles $(30°, 60°, 90°)$, and therefore sides $(x, \sqrt{3}x, 2x)$. This gives $x + \sqrt{3}x + 2x = 1$, so that $x = \frac{1}{6}\big(3 - \sqrt{3}\big)$. The right-angled triangles have side lengths

$$\Big(\tfrac{1}{6}\big(3 - \sqrt{3}\big), \tfrac{1}{2}\big(\sqrt{3} - 1\big), \tfrac{1}{3}\big(3 - \sqrt{3}\big)\Big).$$

Each right-angled triangle has area

$$\begin{aligned} A &= \tfrac{1}{2} \cdot \tfrac{1}{6}\big(3 - \sqrt{3}\big) \cdot \tfrac{1}{2}\big(\sqrt{3} - 1\big) \\ &= \tfrac{1}{12}\big(2\sqrt{3} - 3\big). \end{aligned}$$

The area in common to both equilateral triangles is given by the area of an equilateral triangle minus three of these right-angled triangles. This is

$$\tfrac{\sqrt{3}}{4} - 3 \cdot \tfrac{1}{12}\big(2\sqrt{3} - 3\big) = \tfrac{1}{4}\big(3 - \sqrt{3}\big).$$

So, the ratio between the area of an equilateral triangle and the area in common to both is

$$\tfrac{\sqrt{3}}{4} : \tfrac{1}{4}\big(3 - \sqrt{3}\big).$$

This simplifies to $1 : \sqrt{3} - 1$. So, re-scaling the equilateral triangles to have area 1, the area in common to both is $\sqrt{3} - 1$, as required.

4703. Using a double-angle formula,

$$\cos^2 x \equiv \tfrac{1}{2}(\cos 2x + 1).$$

For parts, let $u = x$ and $v' = \frac{1}{2}(\cos 2x + 1)$, giving $u' = 1$ and $v = \frac{1}{4}\sin 2x + \frac{1}{2}x$:

$$\begin{aligned} &\int x \cdot \tfrac{1}{2}(\cos 2x + 1)\, dx \\ &= \tfrac{1}{4}\sin 2x + \tfrac{1}{2}x^2 - \int \tfrac{1}{4}\sin 2x + \tfrac{1}{2}x\, dx \\ &= \tfrac{1}{4}\sin 2x + \tfrac{1}{8}\cos 2x + \tfrac{1}{4}x^2 + c. \end{aligned}$$

So, setting up the definite integral,

$$\begin{aligned} &\int_0^\pi x \cos^2 x\, dx \\ &= \Big[\tfrac{1}{4}\sin 2x + \tfrac{1}{8}\cos 2x + \tfrac{1}{4}x^2\Big]_0^\pi \\ &= \Big(\tfrac{1}{8} + \tfrac{\pi^2}{4}\Big) - \big(\tfrac{1}{8}\big) \\ &= \tfrac{\pi^2}{4}, \text{ as required.} \end{aligned}$$

4704. By symmetry, the sphere is in contact with three faces. The reaction forces act perpendicular to these. Consider the forces on the sphere. Rotate the scenario so that the reaction forces are $-R\mathbf{i}$, $-R\mathbf{j}$ and $-R\mathbf{k}$. Since the sphere is in equilibrium,

$$\mathbf{W} = R(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Taking the magnitude, $W = \sqrt{3}R$. By NIII, the reaction force exerted by the sphere on each face of the cube is

$$R = \tfrac{\sqrt{3}}{3}W.$$

4705. Expanding with compound-angle formulae,

$$\begin{aligned} &\sin(2x + 1) = \cos(2x - 1) \\ \Longrightarrow\ &\sin 2x \cos 1 + \cos 2x \sin 1 \\ &\qquad = \cos 2x \cos 1 + \sin 2x \sin 1 \\ \Longrightarrow\ &\sin 2x(\cos 1 - \sin 1) = \cos 2x(\cos 1 - \sin 1) \\ \Longrightarrow\ &\tan 2x = 1 \\ \Longrightarrow\ &2x = \frac{(1 + 4n)\pi}{4} \\ \Longrightarrow\ &x = \frac{(1 + 4n)\pi}{8} \text{ for } n \in \mathbb{Z}, \text{ as required.} \end{aligned}$$

4706. The squared distance between (x, y) and $(1, 1)$ is

$$d^2 = (x - 1)^2 + (y - 1)^2.$$

Substituting for y, this is

$$\begin{aligned} d^2 &= (x - 1)^2 + \Big(\big(1 - \sqrt{x}\big)^2 - 1\Big)^2 \\ &\equiv (x - 1)^2 + \big(x - 2\sqrt{x}\big)^2 \\ &\equiv x^2 - 2x + 1 + x^2 - 4x^{\frac{3}{2}} + 4x \\ &\equiv 2x^2 - 4x^{\frac{3}{2}} + 4x + 1. \end{aligned}$$

To maximise this, we set the derivative to zero:

$$\begin{aligned} &4x - 6x^{\frac{1}{2}} + 4 = 0 \\ \Longrightarrow\ &\big(2x^{\frac{1}{2}} - 1\big)\big(x^{\frac{1}{2}} - 2\big) = 0 \\ \Longrightarrow\ &x^{\frac{1}{2}} = 2, \tfrac{1}{2} \\ \Longrightarrow\ &x = 4, \tfrac{1}{4}. \end{aligned}$$

The former doesn't produce points on the curve. The latter gives $(1/4, 1/4)$, at a squared distance of $9/8 > 1$. Also, at the extreme points of the curve, which are $(1, 0)$ and $(0, 1)$, the distance is 1. Hence, the distance between the curve and the point $(1, 1)$ is never less than 1. □

4707. The second terms are $\frac{1}{2}(a+b)$ in the AP and $\sqrt{ab}$ in the GP. We prove the result as follows:

$$\begin{aligned}&(a-b)^2 \geq 0\\ \Longrightarrow\ & a^2-2ab+b^2 \geq 0\\ \Longrightarrow\ & a^2+2ab+b^2 \geq 4ab\\ \Longrightarrow\ & (a+b)^2 \geq 4ab.\end{aligned}$$

Since a and b are positive,

$$\begin{aligned}&a+b \geq 2\sqrt{ab}\\ \Longrightarrow\ & \frac{a+b}{2} \geq \sqrt{ab}, \text{ as required.}\end{aligned}$$

NOTA BENE

This is the AM-GM (arithmetic mean-geometric mean) inequality. Equality holds if and only if $a=b$. In the language of this question, that is if both sequences are constant.

4708. Call $\frac{1}{41}$ of the original integral I:

$$I = \int e^{4x}\sin 5x\,dx.$$

We proceed by the tabular integration method:

Signs	Derivatives	Integrals
$+$	$\sin 5x$	e^{4x}
$-$	$5\cos 5x$	$\frac{1}{4}e^{4x}$
$+$	$-25\sin 5x$	$\frac{1}{16}e^{4x}$

This gives

$$\begin{aligned}&I = \tfrac{1}{4}e^{4x}\sin 5x - \tfrac{5}{16}e^{4x} - \tfrac{25}{16}I\\ \Longrightarrow\ & 41I = 4e^{4x} - 5e^{4x}.\end{aligned}$$

With a constant of integration,

$$\begin{aligned}&\int 41e^{4x}\sin 5x\,dx\\ &= e^{4x}(4\sin 5x - 5\cos 5x) + c, \text{ as required.}\end{aligned}$$

4709. A stationary point on the x axis is a double root. So, the quartic must be expressible as

$$\begin{aligned}&4x^4+4x^3+kx^2-2x+1\\ &\equiv 4(x-a)^2(x-b)^2\\ &\equiv 4(x^2-2ax+a^2)(x^2-2bx+b^2).\end{aligned}$$

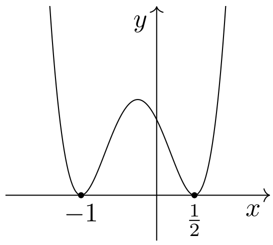
Substituting $x=0$ gives $4a^2b^2=1$, so $b=\pm\frac{1}{2a}$. Equating coefficients of x,

$$\begin{aligned}&-8ab^2-8a^2b=-2\\ \Longrightarrow\ & 4ab^2+4a^2b-1=0.\end{aligned}$$

Substituting $b=\pm\frac{1}{2a}$,

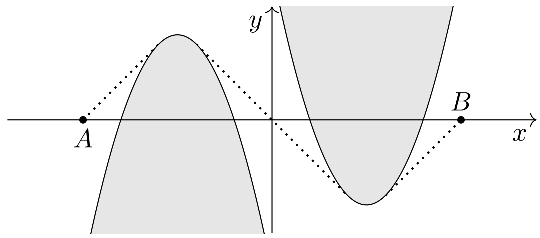
$$\begin{aligned}&\tfrac{1}{a}\pm 2a-1=0\\ \Longrightarrow\ & \pm 2a^2-a+1=0.\end{aligned}$$

The positive version has $\Delta<0$, so no real roots. The negative version gives $a=-1$ or $a={}^1/_2$. These are the x coordinates of the SPs:



The value of k is -3.

4710. The entire map has rotational symmetry around the origin. So, the shortest pass must pass through O. Adding axes, the diagram is



The central section of path has equation $y=kx$. We require this to be tangential to the parabola $y=(x-1)(x-4)$. For intersections,

$$\begin{aligned}&(x-1)(x-4)-kx=0\\ \Longrightarrow\ & x^2-(5+k)x+4=0.\end{aligned}$$

We need this to have exactly one root. Setting the discriminant $\Delta=0$,

$$\begin{aligned}&(5+k)^2-16=0\\ \Longrightarrow\ & (5+k)=\pm 4\\ \Longrightarrow\ & k=-1,-9.\end{aligned}$$

The root $k=-9$ is tangential to the parabola for negative x and large y. This is not the root we require. The relevant path is $y=x$. This gives an intersection at $(2,-2)$. So, the vector between the points of tangency is $4\mathbf{i}+4\mathbf{j}$. The distance, therefore, is $4\sqrt{2}$ km, as required.

4711. For $v \ll c$, powers of ${}^{v^2}/_{c^2}$ greater than 2 can be neglected. Setting aside mc^2 for now, consider the following expansion:

$$\begin{aligned}&\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\\ &\approx 1+(-\tfrac{1}{2})\left(-\frac{v^2}{c^2}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{v^2}{c^2}\right)^2\\ &= 1+\frac{v^2}{2c^2}+\frac{3v^4}{8c^4}.\end{aligned}$$

For the energy, we multiply by mc^2. This gives

$$E \approx mc^2+\frac{mv^2}{2}+\frac{3mv^4}{8c^2}.$$

— Nota Bene —

① The term $E_0 = mc^2$ is the *rest energy* of a massive particle. This is not modelled at all by Newtonian theory. It describes the energy released in nuclear reactions.

② The term $E_1 = \frac{1}{2}mv^2$ is *kinetic energy* in the Newtonian model, a first approximation to special relativity, which holds for small v.

③ The term $E_2 = \frac{3mv^4}{8c^2}$ is *kinetic energy* in the first post-Newtonian approximation. This is non-negligible as v gets large.

Terms E_2 and the neglected $E_3, E_4, ...$ represent the breakdown of Newtonian theory as particles reach speeds comparable to that of light.

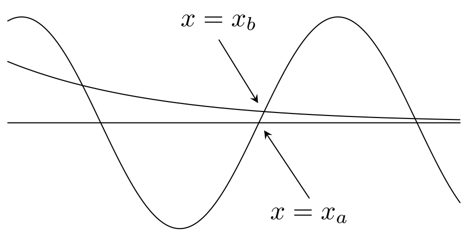
4712. Dividing both sides by e^x,

$$1 + e^{-x}\cos x = \sin^2 x.$$

By the Pythagorean identity, this is

$$\begin{aligned} &\cos^2 x + e^{-x}\cos x = 0 \\ \Longrightarrow\ &\cos x(\cos x + e^{-x}) = 0 \\ \Longrightarrow\ &\cos x = 0 \text{ or } \cos x = e^{-x}. \end{aligned}$$

Consider $y = \cos x$, $y = 0$ and $y = e^{-x}$. For large x, the behaviour is as shown (not to scale):



For any positive root of $\cos x = 0$, there is a root of $\cos x = e^{-x}$ just before or just after it. As $e^{-x} \to 0$, these get closer together, producing pairs of roots $x_a < x_b$ with $x_b - x_a \to 0$. □

4713. Completing the square,

$$9x^2 + 6x + 2 = (3x + 1)^2 + 1.$$

Let $3x + 1 = \tan\theta$, which gives $3\,dx = \sec^2\theta\,d\theta$. As $x \to \pm\infty$, $\theta \to \pm\pi/2$; these are the new limits. Enacting the substitution,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{3}{(3x+1)^2 + 1}\,dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\tan^2\theta + 1}\sec^2\theta\,d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,d\theta \\ &= \big[\theta\big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \pi, \text{ as required.} \end{aligned}$$

4714. The first equation is quadratic in y:

$$\begin{aligned} & 2y - 1 = (x-1)(y^2 - 2y + 1) \\ \Longrightarrow\ & (x-1)y^2 - 2xy + x = 0 \\ \Longrightarrow\ & y = \frac{2x \pm \sqrt{4x^2 - 4x(x-1)}}{2(x-1)} \\ & \equiv \frac{x \pm \sqrt{x}}{x - 1} \\ & \equiv \frac{\sqrt{x}\,(\sqrt{x} \pm 1)}{(\sqrt{x}+1)\,(\sqrt{x}-1)}. \end{aligned}$$

Dividing top and bottom by $(\sqrt{x} \mp 1)$ gives

$$y = \frac{\sqrt{x}}{\sqrt{x} \pm 1}.$$

So, the required implication holds.

4715. Assume, for a contradiction, that there are finitely many prime numbers. Call them

$$p_1 < p_2 < ... < p_n.$$

Let $P = p_1p_2...p_n$ and consider $P + 1$. Since P is divisible by all of the prime numbers, $P + 1$ is divisible by none of them. Hence, $P + 1$ has no prime factors other than itself, and must be prime. But $P + 1$ is larger than p_n, so cannot be in the list. This is a contradiction. Therefore, there are infinitely many prime numbers. □

4716. (a) We know that $\mathrm{g}''(x) - \mathrm{h}''(x) = 0$ for all x. Integrating this twice,

$$\mathrm{g}(x) - \mathrm{h}(x) = ax + b.$$

So, $\mathrm{g}(x) - \mathrm{h}(x)$ is linear in x.

(b) By the quotient rule,

$$\begin{aligned} & y = \frac{\mathrm{g}(x)}{\mathrm{h}(x)} \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{\mathrm{g}'(x)\,\mathrm{h}(x) - \mathrm{g}(x)\,\mathrm{h}'(x)}{\mathrm{h}(x)^2}. \end{aligned}$$

Because $\mathrm{h}(x) > 0$, this is zero exactly when its numerator is zero:

$$\mathrm{g}'(x)\,\mathrm{h}(x) - \mathrm{g}(x)\,\mathrm{h}'(x) = 0.$$

Since $\mathrm{g}(x) - \mathrm{h}(x)$ is constant, we can substitute $\mathrm{g}(x) = \mathrm{h}(x) + k$ and $\mathrm{g}'(x) = \mathrm{h}'(x)$, giving

$$\begin{aligned} & \mathrm{h}'(x)\,\mathrm{h}(x) - (\mathrm{h}(x) + k)\,\mathrm{h}'(x) = 0 \\ \Longleftrightarrow\ & -k\,\mathrm{h}'(x) = 0. \end{aligned}$$

This always has at least one root, because $\mathrm{h}'(x)$, as the derivative of a quartic, is cubic. So, there is at least one stationary value of

$$y = \frac{\mathrm{g}(x)}{\mathrm{h}(x)}.$$

4717. (a) By Pythagoras,

$$\begin{aligned}|AX| &= \sqrt{(\cos\theta+1)^2+\sin^2\theta}\\ &\equiv \sqrt{\cos^2\theta+2\cos\theta+1+\sin^2\theta}\\ &\equiv \sqrt{2+2\cos\theta}, \text{ as required.}\end{aligned}$$

(b) Using the given identity,

$$|AX| = 2\cos\tfrac{1}{2}\theta.$$

The other lengths can be found together, the only difference being a $\mp$:

$$\begin{aligned}|BX|,|CX| &= \sqrt{(\cos\theta-{}^1\!/_2)^2+(\sin\theta\mp{}^{\sqrt{3}}\!/_2)^2}\\ &\equiv \sqrt{2-\cos\theta\mp\sqrt{3}\sin\theta}.\end{aligned}$$

In harmonic form, this is

$$\sqrt{2+2\cos\left(x\pm\tfrac{2\pi}{3}\right)}.$$

Using the given identity again gives

$$2\cos\left(\tfrac{1}{2}\theta\pm\tfrac{\pi}{3}\right).$$

Using a compound-angle formula,

$$|BX|,|CX| = \cos\tfrac{1}{2}\theta\mp\sqrt{3}\sin\tfrac{1}{2}\theta.$$

Adding these, the sines cancel:

$$|BX|+|CX| = 2\cos\tfrac{1}{2}\theta.$$

So, $|AX| = |BX|+|CX|$, as required.

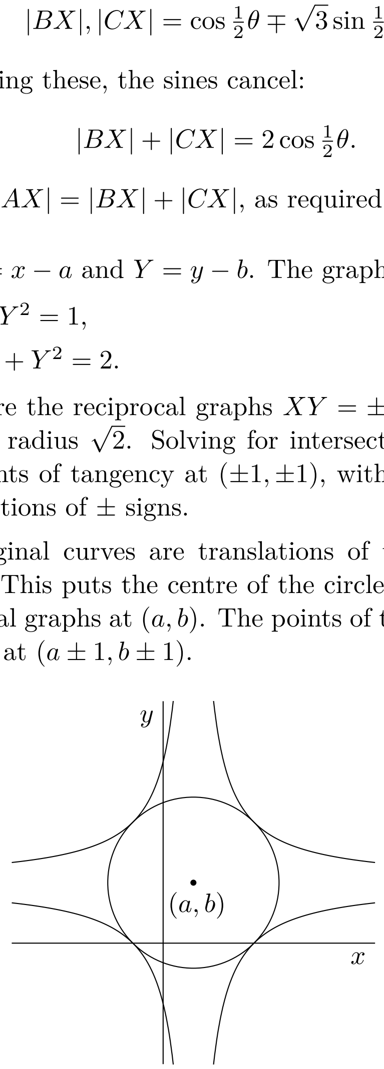
4718. Let $X = x-a$ and $Y = y-b$. The graphs are

① $X^2Y^2 = 1$,

② $X^2+Y^2 = 2$.

These are the reciprocal graphs $XY = \pm 1$, and a circle of radius $\sqrt{2}$. Solving for intersections, we find points of tangency at $(\pm 1, \pm 1)$, with all four combinations of $\pm$ signs.

The original curves are translations of these by $a\mathbf{i}+b\mathbf{j}$. This puts the centre of the circle and the reciprocal graphs at (a,b). The points of tangency are now at $(a\pm 1, b\pm 1)$.



4719. Expanding the RHS,

$$\begin{aligned}&2+\sqrt{2}\sin\left(x-\tfrac{\pi}{4}\right)\\ \equiv\ &2+\sqrt{2}\big(\sin x\cos\tfrac{\pi}{4}-\cos x\sin\tfrac{\pi}{4}\big)\\ \equiv\ &2+\sin x-\cos x.\end{aligned}$$

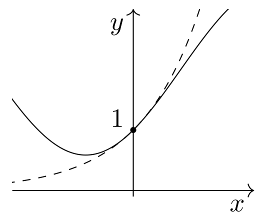
For small x, $\sin x\approx x$ and $\cos x\approx 1-\frac{1}{2}x^2$. So, the RHS is approximately

$$\begin{aligned}&2+x-\left(1-\tfrac{1}{2}x^2\right)\\ \equiv\ &1+x+\tfrac{1}{2}x^2.\end{aligned}$$

This is the given quadratic approximation to e^x. So, for x close to zero, $e^x\approx 2+\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$.

— Nota Bene —

The graphs, with $y = e^x$ dashed, are:



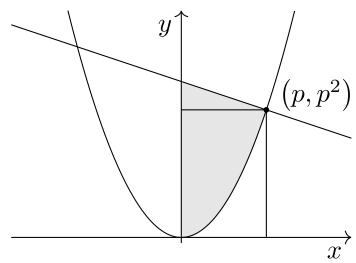
4720. Consider the positive quadrant only, where $|xy|$ is simply xy. The arc length is $\frac{\pi}{2}$. Let θ be the usual angle anticlockwise from the positive x axis, giving $xy = \cos\theta\sin\theta$. This is $\frac{1}{2}\sin 2\theta$. So, the relevant calculation is

$$\begin{aligned}&\frac{1}{\pi/2}\int_0^{\frac{\pi}{2}}\tfrac{1}{2}\sin 2\theta\,d\theta\\ =\ &\tfrac{2}{\pi}\Big[-\tfrac{1}{4}\cos 2\theta\Big]_0^{\frac{\pi}{2}}\\ =\ &\tfrac{2}{\pi}\Big(\tfrac{1}{4}-\big(-\tfrac{1}{4}\big)\Big)\\ =\ &\tfrac{1}{\pi}, \text{ as required.}\end{aligned}$$

4721. The equation of the normal at (p, p^2) is

$$y-p^2 = -\tfrac{1}{2p}(x-p).$$

Setting $x = 0$, the y intercept is at $y = p^2+\frac{1}{2}$. The relevant area is that of a rectangle plus that of a triangle, minus that under the curve.



$$\begin{aligned}A &= A_{\text{rect}}+A_{\text{tri}}-A_{\text{curve}}\\ &= p^3+\tfrac{1}{4}p-\int_0^p x^2\,dx\\ &\equiv p^3+\tfrac{1}{4}p-\tfrac{1}{3}p^3\\ &\equiv \tfrac{2}{3}p^3+\tfrac{1}{4}p, \text{ as required.}\end{aligned}$$

4722. The derivatives with respect to t are $\dot{x} = -2\sin 2t$ and $\dot{y} = 12\sin^2 t\cos t$. By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{12\sin^2 t\cos t}{-2\sin 2t} \\ &\equiv \frac{6\sin^2 t\cos t}{-2\sin t\cos t} \\ &\equiv -3\sin t.\end{aligned}$$

The point P has (primary) parameter $t = \frac{\pi}{6}$. So, the gradient at P is $-\frac{3}{2}$. The tangent at P is

$$\begin{aligned}& y - \tfrac{1}{2} = -\tfrac{3}{2}\big(x - \tfrac{1}{2}\big) \\ \Longrightarrow\ & y = -\tfrac{3}{2}x + \tfrac{5}{4}.\end{aligned}$$

This intersects the curve where

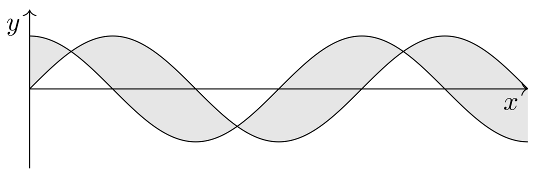
$$\begin{aligned}& 4\sin^3 t = -\tfrac{3}{2}\cos 2t + \tfrac{5}{4} \\ \Longrightarrow\ & 4\sin^3 t = -\tfrac{3}{2}\big(1 - 2\sin^2 t\big) + \tfrac{5}{4} \\ \Longrightarrow\ & 16\sin^3 t - 12\sin^2 t + 1 = 0 \\ \Longrightarrow\ & (2\sin t - 1)^2(4\sin t + 1) = 0.\end{aligned}$$

The squared factor corresponds to P. Hence, at Q, $\sin t = -\frac{1}{4}$. This gives coordinates $(7/8, -1/16)$, as required.

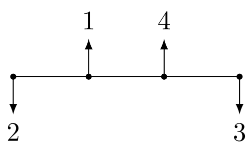
4723. Factorising, the inequality is

$$\begin{aligned}& y^2 + \sin x\cos x - y\sin x - y\cos x \le 0 \\ \Longleftrightarrow\ & (y - \sin x)(y - \cos x) \le 0.\end{aligned}$$

Such a product is non-positive iff exactly one of its factors is non-positive. So, points (x, y) which satisfy this are between $y = \sin x$ and $y = \cos x$:



4724. It is possible, as the following example shows:



The resultant force is zero, so equilibrium is not broken by translation. Also, the resultant moment around $x = 1$ is

$$1 \times 1 + 4 \times 2 - 3 \times 3 = 0.$$

So, neither does the object rotate.

4725. Writing $\sin^2 x \equiv 1 - \cos^2 x$,

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \sin^3 x\, dx \\ = &\int_0^{\frac{\pi}{2}} \sin x - \sin x\cos^2 x\, dx.\end{aligned}$$

We integrate the second term by inspection, giving

$$\begin{aligned}&\Big[-\cos x + \tfrac{1}{3}\cos^3 x\Big]_0^{\frac{\pi}{2}} \\ = &\ \big(0\big) - \big(-1 + \tfrac{1}{3}\big) \\ = &\ \tfrac{2}{3}, \text{ as required.}\end{aligned}$$

4726. The circle is $x^2 + y^2 = 10$. We rearrange the first equation to $y = \sqrt[3]{26 - x^3}$. Substituting this in,

$$\begin{aligned}& x^2 + \big(26 - x^3\big)^{\frac{2}{3}} = 10 \\ \Longrightarrow\ & x^2 + \big(26 - x^3\big)^{\frac{2}{3}} - 10 = 0.\end{aligned}$$

The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{x_n^2 + \big(26 - x_n^3\big)^{\frac{2}{3}} - 10}{2x_n + \frac{2}{3}\big(26 - x_n^3\big)^{-\frac{1}{3}} \cdot -3x_n^2}.$$

Running this iteration

① with $x_0 = 2$, we get $x_1 = 1.0835$ and then $x_n \to 1.3247$. This gives $y = 2.8714$.
② with $x_0 = -2$, we get $x_1 = -1.3052$ and then $x_n \to -1$. This gives $y = 3$. Testing this, it is an exact solution.

The curves are symmetrical in $y = x$. So, the four points of intersection are $(1.32, 2.87)$ and $(2.87, 1.32)$ to 3sf, and $(-1, 3)$ and $(3, -1)$ exactly.

4727. The derivative is

$$\mathrm{f}'(x) = 1 + x + x^2 + \tfrac{2}{3}x^3 + \tfrac{1}{4}x^4.$$

We need to show that this is positive for all $x \in \mathbb{R}$. Looking for stationary values of the derivative $\mathrm{f}'(x)$, we differentiate again:

$$\mathrm{f}''(x) = 1 + 2x + 2x^2 + x^3.$$

Setting this to zero, we find a root of $\mathrm{f}''(x)$ at $x = -1$. Taking out the relevant factor,

$$\big(1 + x\big)\big(1 + x + x^2\big) = 0.$$

The quadratic factor has $\Delta = -3 < 0$, so has no real roots. Hence, there is only one stationary value of $\mathrm{f}'(x)$, which is $\mathrm{f}'(-1) = \frac{7}{12}$. Since this is positive, $\mathrm{f}'(x)$ must (because it is a polynomial of *even* degree) be positive for all $x \in \mathbb{R}$.

So, f is increasing everywhere, as required.

4728. Multiplying up by the denominators, we require

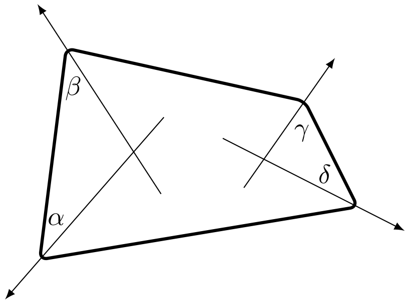
$$x^2 + 2x + 3 \equiv A(x^2 + 1) + Bx.$$

Equating coefficients of x^2, $A = 1$. But, equating the constant terms, $A = 3$. This is a contradiction. Hence, there are no constants $A, B \in \mathbb{R}$ for which this is an identity.

4729. The rope is smooth, so the tension is the same everywhere. By symmetry, therefore, the lines of action must lie along the angle bisectors. Let the half-angles at A, B, C, D be $\alpha, \beta, \gamma, \delta$. We know

$$\alpha + \beta + \gamma + \delta = 180°.$$

Consider two triangles, those formed of A, B and the intersection of their angle bisectors, and C, D and the intersection of their angle bisectors.



These have interior angles

$$(\alpha, \beta, 180° - \alpha - \beta)$$
$$(\gamma, \delta, 180° - \gamma - \delta)$$

The quadrilateral of intersections contains angles $180° - \alpha - \beta$ and $180° - \gamma - \delta$, adding to

$$\begin{aligned} &(180° - \alpha - \beta) + (180° - \gamma - \delta) \\ &= 360° - \alpha - \beta - \gamma - \delta \\ &= 180°. \end{aligned}$$

Hence, according to the relevant circle theorem, the quadrilateral of intersections is cyclic. QED.

4730. (a) Using identities,

$$\begin{aligned} y^2 &= 4\sin^2 t \cos^2 t \\ &\equiv 4(1 - \cos^2 t)\cos^2 t \\ &= 4x^2(1 - x^2). \end{aligned}$$

(b) At a point of self-intersection, two different values of the parameter produce the same (x, y) point. Calling these $s, t \in [0, 2\pi)$,

$$\begin{aligned} &\cos s = \cos t \\ \therefore \quad &s + t = 2\pi. \end{aligned}$$

Substituting this in,

$$\begin{aligned} &\sin 2(2\pi - t) = \sin 2t \\ \Longrightarrow\ &-\sin 2t = \sin 2t \\ \Longrightarrow\ &\sin 2t = 0 \\ \therefore \quad &t = 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}. \end{aligned}$$

The coordinates are

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
(x, y)	$(1, 0)$	$(0, 0)$	$(-1, 0)$	$(0, 0)$.

So, there is one point of self-intersection, which is the origin. This is attained at the primary parameter values $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

4731. (a) Using the product rule,

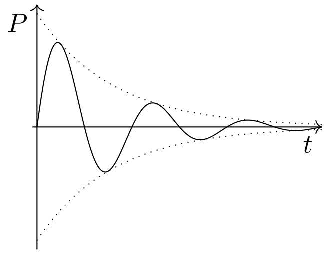
$$\begin{aligned} &P = e^{-2x}\sin kx \\ \Longrightarrow\ &\frac{dP}{dt} = -2e^{-2x}\sin kx + ke^{-2x}\cos kx \\ &\qquad \equiv e^{-2x}\big(-2\sin kx + k\cos kx\big) \\ \Longrightarrow\ &\frac{d^2P}{dt^2} = -2e^{-2x}\big(-2\sin kx + k\cos kx\big) \\ &\qquad + e^{-2x}\big(-2k\cos kx - k^2\sin kx\big) \\ &\qquad \equiv e^{-2x}\Big(\big(4 - k^2\big)\sin kx - 4k\cos kx\Big). \end{aligned}$$

Substituting into the LHS of the DE, there is a common factor of e^{-2x}. This is non-zero. The other factor is

$$\begin{aligned} &\Big(\big(4 - k^2\big)\sin kx - 4k\cos kx\Big) \\ &\qquad + 4\big(-2\sin kx + k\cos kx\big) + 13\sin kx \\ &\equiv \big(9 - k^2\big)\sin kx. \end{aligned}$$

So, if we choose $k = 3$, then the LHS of the DE is identically zero. The relevant population curve is $P = e^{-2x}\sin 3x$.

(b) The population curve is a sinusoidal oscillation around the historical value, whose amplitude decays exponentially:



In the long-term, the oscillations dwindle to zero. Population returns to its historical size.

4732. Let P be the point of tangency for $x > 0$. The x intercept k of the tangent line is minimised if P is the point of inflection of the curve. Looking for this boundary case, we differentiate:

$$\begin{aligned} &y = e^{-\frac{1}{2}x^2} \\ \Longrightarrow\ &\frac{dy}{dx} = -xe^{-\frac{1}{2}x^2} \\ \Longrightarrow\ &\frac{d^2y}{dx^2} = (x^2 - 1)e^{-\frac{1}{2}x^2}. \end{aligned}$$

So, the points of inflection are at $x = \pm 1$. At $x = 1$, the gradient is $m = -e^{-\frac{1}{2}}$. Hence, the equation of the tangent is

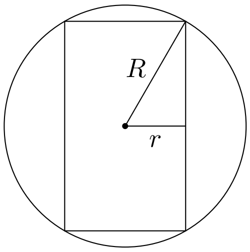
$$y - e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}(x - 1).$$

Substituting $y = 0$,

$$\begin{aligned} &-e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}(x - 1) \\ \Longrightarrow\ &x = 2. \end{aligned}$$

This is the minimal boundary case. There is no upper bound on the value of k. So, the general result, including symmetrical tangents for $x < 0$, is $|k| \geq 2$.

4733. Consider the boundary case, in which the cylinder has maximal volume. It must touch the surface of the sphere. In cross-section, the scenario is



By Pythagoras, the height of the cylinder is

$$h = 2\sqrt{R^2 - r^2}.$$

So, the volume of the cylinder is

$$\begin{aligned} V &= \pi r^2 h \\ &= 2\pi r^2 \sqrt{R^2 - r^2}. \end{aligned}$$

For optimisation, we set the derivative to zero:

$$\begin{aligned} & 4\pi r\sqrt{R^2 - r^2} \\ & \quad + 2\pi r^2 \cdot \tfrac{1}{2}\left(R^2 - r^2\right)^{-\frac{1}{2}} \cdot -2r = 0 \\ \Longrightarrow\ & 2r\left(R^2 - r^2\right) - r^3 = 0 \\ \Longrightarrow\ & 2rR^2 - 3r^3 = 0 \\ \Longrightarrow\ & r = 0 \text{ or } r^2 = \tfrac{2}{3}R^2. \end{aligned}$$

The former is clearly a minimum. So, we use the latter. It is maximal, so the volume must satisfy

$$\begin{aligned} V &\le 2\pi\left(\tfrac{2}{3}R^2\right)\sqrt{R^2 - \tfrac{2}{3}R^2} \\ &= \frac{4\pi R^3}{3\sqrt{3}}, \text{ as required.} \end{aligned}$$

4734. Differentiating by the quotient rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2(x^2-1)^{-\frac{1}{2}} - \left(\sqrt{x^2-1} - 1\right)}{x^2} \\ &\equiv \frac{x^2 - (x^2-1) + \sqrt{x^2-1}}{x^2\sqrt{x^2-1}} \\ &\equiv \frac{\sqrt{x^2-1} + 1}{x^2\sqrt{x^2-1}}. \end{aligned}$$

Setting this to $\frac{3}{10}$,

$$\begin{aligned} & \frac{\sqrt{x^2-1}+1}{x^2\sqrt{x^2-1}} = \frac{3}{10} \\ \Longrightarrow\ & 10\sqrt{x^2-1} + 10 = 3x^2\sqrt{x^2-1} \\ \Longrightarrow\ & (3x^2 - 10)\sqrt{x^2-1} = 10 \\ \Longrightarrow\ & 9x^6 - 69x^4 + 160x^2 - 200 = 0. \end{aligned}$$

This is a cubic in x^2. Using a polynomial solver, $x^2 = 5$. This gives $x = \pm\sqrt{5}$. So, the coordinates at which the gradient is $\frac{3}{10}$ are

$$\left(\pm\sqrt{5}, \pm\tfrac{1}{\sqrt{5}}\right).$$

4735. Since the graph $y = \mathrm{h}(x)$ is symmetrical around $x = 0$, the mean of the roots is 0. Hence, we can express them as $\{-3k, -k, k, 3k\}$, where k is half the common difference. According to the factor theorem,

$$\begin{aligned} \mathrm{h}(x) &= (x+3k)(x+k)(x-k)(x-3k) \\ &\equiv \left(x^2 - 9k^2\right)\left(x^2 - k^2\right) \\ &= x^4 - 10k^2x^2 + 9k^4, \text{ as required.} \end{aligned}$$

4736. To simplify $\sin(\arctan x)$, let $x = \tan y$. Then

$$\begin{aligned} \sin(\arctan x) &= \sin y \\ &= \frac{1}{\sqrt{\operatorname{cosec}^2 y}} \\ &\equiv \frac{1}{\sqrt{1 + \cot^2 y}} \\ &= \frac{1}{\sqrt{1 + \frac{1}{x^2}}} \\ &\equiv \frac{x}{\sqrt{x^2+1}}. \end{aligned}$$

Also,

$$\begin{aligned} \cos(\arctan x) &= \cos y \\ &= \frac{1}{\sqrt{\sec^2 y}} \\ &\equiv \frac{1}{\sqrt{1 + \tan^2 y}} \\ &= \frac{1}{\sqrt{1+x^2}}. \end{aligned}$$

Substituting $x = \frac{1}{2}$,

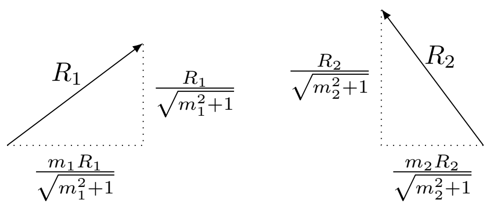
$$\begin{aligned} \sin\left(\arctan \tfrac{1}{2}\right) &= \frac{1/2}{\sqrt{5/4}} = \tfrac{1}{\sqrt{5}}, \\ \cos\left(\arctan \tfrac{1}{2}\right) &= \frac{1}{\sqrt{5/4}} = \tfrac{2}{\sqrt{5}}. \end{aligned}$$

Multiplying these and a factor of 2,

$$\sin\left(2\arctan\tfrac{1}{2}\right) = 2 \cdot \tfrac{1}{\sqrt{5}} \cdot \tfrac{2}{\sqrt{5}} = \tfrac{4}{5}.$$

So, $2\arctan\frac{1}{2} = \arcsin\frac{4}{5}$, as required.

4737. The gradients of the lines are $-m_1$ and m_2, so the gradients of the reaction forces are $1/m_1$ and $-1/m_2$. The forces and their components are



Horizontal equilibrium gives

$$\frac{m_1 R_1}{\sqrt{m_1^2+1}} - \frac{m_2 R_2}{\sqrt{m_2^2+1}} = 0.$$

Vertical equilibrium gives

$$\frac{R_1}{\sqrt{m_1^2+1}} + \frac{R_2}{\sqrt{m_2^2+1}} = W.$$

Adding $\frac{1}{m_2}$ times the horizontal to the vertical,

$$\begin{aligned}&\frac{R_1}{\sqrt{m_1^2+1}} + \frac{\frac{m_1}{m_2}R_1}{\sqrt{m_1^2+1}} = W\\ \Longrightarrow\ & R_1\frac{1+\frac{m_1}{m_2}}{\sqrt{m_1^2+1}} = W\\ \Longrightarrow\ & R_1 = \frac{\sqrt{m_1^2+1}}{1+\frac{m_1}{m_2}}W.\end{aligned}$$

Multiplying top and bottom by m_2,

$$R_1 = \frac{m_2\sqrt{m_1^2+1}}{m_1+m_2}W.$$

The component diagrams are symmetrical, so the corresponding result is

$$R_2 = \frac{m_1\sqrt{m_2^2+1}}{m_1+m_2}W.$$

4738. Separating the variables,

$$\int \cos^2 3y\,dy = \int \sqrt{x^4+2x^2}\,dx.$$

Using a double-angle formula, the LHS is

$$\begin{aligned}&\int \tfrac{1}{2}(1+\cos 6y)\,dy\\ &= \tfrac{1}{2}y + \tfrac{1}{12}\sin 6y + c_1.\end{aligned}$$

Taking out a factor of $\sqrt{x^2}$, the RHS can then be integrated by inspection:

$$\begin{aligned}&\int x\sqrt{x^2+2}\,dx\\ &= \tfrac{1}{2}\int 2x\sqrt{x^2+2}\,dx\\ &= \tfrac{1}{2}\cdot\tfrac{2}{3}(x^2+2)^{\frac{3}{2}} + c_2\\ &\equiv \tfrac{1}{3}(x^2+2)^{\frac{3}{2}} + c_2.\end{aligned}$$

Multiplying LHS and RHS by 12 and combining the constants of integration, the general solution is as required:

$$6y + \sin 6y = 4(x^2+2)^{\frac{3}{2}} + c.$$

4739. Using a compound-angle formula, the RHS is

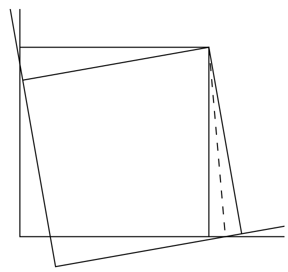
$$\begin{aligned}&10\cos\big(\theta + \arctan\tfrac{3}{4}\big)\\ \equiv\ & 10\cos\theta\cos\big(\arctan\tfrac{3}{4}\big) - 10\sin\theta\sin\big(\arctan\tfrac{3}{4}\big)\\ \equiv\ & 8\cos\theta - 6\sin\theta.\end{aligned}$$

Using a double-angle formula, the full equation is

$$\begin{aligned}&6\sin\theta\cos\theta - 8 = 8\cos\theta - 6\sin\theta\\ \Longrightarrow\ & 3\sin\theta\cos\theta + 3\sin\theta - 4\cos\theta - 4 = 0\\ \Longrightarrow\ & (3\sin\theta - 4)(\cos\theta + 1) = 0.\end{aligned}$$

The first factor has no roots, as $\frac{4}{3} > 1$. The second factor gives $\cos\theta = -1$, so $\theta = \pi$.

4740. The line (dashed below) to the point of intersection near the midpoints of the sides bisects the angle of rotation θ.



The kite of which it is a diagonal has long sides of length $\frac{1}{2}$ and short sides of length $\frac{1}{2}\tan\frac{1}{2}\theta$. We can approximate this with $\frac{1}{4}\theta$. Subtracting this from 1, the long non-hypotenuses of the triangles have approximate length $\frac{1}{2} - \frac{1}{4}\theta$. This gives their short sides as

$$\tan\theta\left(\tfrac{1}{2} - \tfrac{1}{4}\theta\right) \approx \theta\left(\tfrac{1}{2} - \tfrac{1}{4}\theta\right).$$

So, using $\frac{1}{2}bh$, the total area of the triangles is

$$\begin{aligned}A &\approx 8\times\tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{4}\theta\right)\cdot\theta\left(\tfrac{1}{2} - \tfrac{1}{4}\theta\right)\\ &= \theta - \theta^2 + \tfrac{1}{4}\theta^3.\end{aligned}$$

Neglecting the term in θ^3,

$$A \approx \theta(1-\theta), \text{ as required.}$$

4741. The restricted possibility space consists of all strictly increasing sequences of four scores. There is exactly one increasing sequence for every set of four scores. Therefore, the restricted possibility space contains ${}^6\mathrm{C}_4 = 15$ outcomes. There are ${}^5\mathrm{C}_3 = 10$ of these which contain a six. So, the probability is $2/3$.

4742. The equation of the trajectory with initial speed u and angle of projection θ is

$$\begin{aligned}& y = x\tan\theta - x^2\frac{g}{2u^2}(\tan^2\theta + 1)\\ \Longrightarrow\ & gx^2\tan^2\theta - 2u^2x\tan\theta + gx^2 - 2u^2y = 0.\end{aligned}$$

This is a quadratic in $\tan\theta$. For (x, y) points in the shaded region, there are two possible angles of projection. For those outside, there are none.

So, on the parabola of safety, there is one. Setting $\Delta = 0$,

$$4u^4x^2 - 4gx^2(gx^2 - 2u^2y) = 0$$
$$\Longrightarrow x^2(u^4 - g^2x^2 - 2gu^2y) = 0.$$

The value $x = 0$ corresponds only to points on the y axis. So, we can divide through by $4x^2$, giving

$$u^4 - g^2x^2 - 2gu^2y$$
$$\Longrightarrow y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}.$$

4743. (a) Using the factorial definition of ${}^n\mathrm{C}_r$,

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!}{(k+2)!(n-k-2)!}.$$

Dividing by $n!$ (it cannot be zero),

$$\frac{1}{k!(n-k)!} + \frac{1}{(k+1)!(n-k-1)!} = \frac{1}{(k+2)!(n-k-2)!}.$$

Multiplying by $(k+2)!(n-k)!$,

$$(k+1)(k+2) + (k+2)(n-k) = (n-k)(n-k-1)$$
$$\Longrightarrow n^2 - (3k+3)n + k^2 - 2 = 0.$$

The quadratic formula gives

$$n = \frac{3k+3 \pm \sqrt{(3k+3)^2 - 4(k^2-2)}}{2} \equiv \frac{3k+3 \pm \sqrt{5k^2+18k+17}}{2}.$$

(b) For both n and k to be natural numbers, the discriminant must be a square. Testing the values of k in $\Delta = 5k^2 + 18k + 17$, we find that only $k = 4$ generates a square: 169. This gives

$$n = \frac{15 \pm 13}{2} = 1, 14.$$

We reject the former, as ${}^n\mathrm{C}_k$ is undefined for $n < k$. So, $k = 4$ and $n = 14$:

$${}^{14}\mathrm{C}_4 + {}^{14}\mathrm{C}_5 = 1001 + 2002 = 3003 = {}^{14}\mathrm{C}_6.$$

4744. Let $u = \sqrt{x} + 1$. Then $du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$, which we can rewrite as $dx = 2(u-1)\,du$. The new limits are $u = 1$ to $u = 2$.

$$\int_0^1 \frac{1}{\sqrt{x}+1}\,dx = \int_1^2 \frac{1}{u} \cdot 2(u-1)\,du.$$

Multiplying the integrand out, this is

$$\int_1^2 2 - \frac{2}{u}\,du$$
$$= \Big[2u - 2\ln|u|\Big]_1^2$$
$$= \big(4 - 2\ln 2\big) - \big(2 - 2\ln 1\big)$$
$$= 2 - \ln 4, \text{ as required.}$$

4745. The upper quarter circle has centre $(0, -1)$ and radius $\sqrt{2}$. So, for points on it,

$$x^2 + (y+1)^2 = 2$$
$$\therefore \quad y = \sqrt{2 - x^2} - 1.$$

The area of the shaded rectangle is

$$A = 4xy = 4x\sqrt{2-x^2} - 4x.$$

Setting the derivative to zero,

$$4\sqrt{2-x^2} + 4x \cdot \tfrac{1}{2}\big(2-x^2\big)^{-\frac{1}{2}}(-2x) - 4 = 0$$
$$\Longrightarrow \sqrt{2-x^2} - x^2\big(2-x^2\big)^{-\frac{1}{2}} - 1 = 0.$$

Multiplying by $\sqrt{2-x^2}$,

$$\big(2-x^2\big) - x^2 - \sqrt{2-x^2} = 0$$
$$\Longrightarrow 2 - 2x^2 = \sqrt{2-x^2}$$
$$\Longrightarrow 4 - 8x^2 + 4x^4 = 2 - x^2$$
$$\Longrightarrow 4x^4 - 7x^2 + 2 = 0$$
$$\Longrightarrow x^2 = \tfrac{7 \pm \sqrt{17}}{8}.$$

Both roots are positive. The larger gives a value of x^2 greater than 1, which isn't possible. So, the required value is

$$x = \sqrt{\tfrac{7-\sqrt{17}}{8}}.$$

4746. From the standard definition of the derivative,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}.$$

To use the expansion $a^3 - b^3 \equiv (a-b)\big(a^2 + ab + b^2\big)$, we multiply top and bottom by

$$(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{1}{3}}.$$

This gives

$$\lim_{h\to 0} \frac{(x+h) - x}{h\big((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{1}{3}}\big)}$$
$$\equiv \lim_{h\to 0} \frac{1}{\big((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{1}{3}}\big)}.$$

At this point we can take the limit:

$$\frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}} = \tfrac{1}{3}x^{-\frac{2}{3}}, \text{ as required.}$$

4747. The proposed solution and its derivatives are

$$\begin{aligned} & y^{(0)} = xe^x \\ \Longrightarrow\ & y^{(1)} = (x+1)e^x \\ \Longrightarrow\ & y^{(2)} = (x+2)e^x \\ \Longrightarrow\ & y^{(3)} = (x+3)e^x \\ \Longrightarrow\ & y^{(4)} = (x+4)e^x. \end{aligned}$$

Substituting into the LHS of the DE,

$$\begin{aligned} & y^{(4)} - 2y^{(3)} + 2y^{(2)} - 2y^{(1)} + y^{(0)} \\ = & \big(x+4-2(x+3)+2(x+2)-2(x+1)+x\big)e^x \\ \equiv & \big(0\big)e^x \\ \equiv & \ 0. \end{aligned}$$

The proposed solution satisfies the DE.

— Nota Bene —

The notation $y^{(n)}$ for the nth derivative comes in handy when expressing general differentiation formulae. For example, the general result in this question, with $y = xe^x$, is

$$y^{(n)} = (x+n)e^x.$$

4748. Setting $\frac{dy}{dx} = 0$,

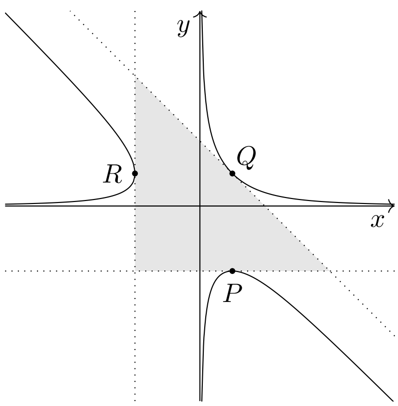
$$\begin{aligned} & 2xy + y^2 = 0 \\ \Longrightarrow\ & y = 0 \text{ or } y = -2x. \end{aligned}$$

There are no points on the curve at which $y = 0$. Instead substituting $y = -2x$ in,

$$\begin{aligned} & x^2(-2x) + x(-2x)^2 = 2 \\ \Longrightarrow\ & x = 1. \end{aligned}$$

At $x = 1$, $y = -2$ or 1. Testing the gradients, we find that the tangent is parallel to the x axis at $(1, -2)$ and has gradient 1 at $(1, 1)$. So, the former is P and the latter is R.

The problem is symmetrical in $y = x$, so Q must be $(-2, 1)$. Sketching the curve and the tangents:



The vertices of the triangle are at $(-2, -2)$, $(-2, 4)$ and $(4, -2)$. Its area is $\frac{1}{2} \cdot 6^2 = 18$, as required.

4749. Consider $x \geq 0$ and $x < 0$ separately:

① On $[0, \infty)$, the inequality is

$$(5-x)(5-x) > 9.$$

For $x \in [0, \infty)$, this gives $x \in [0, 2) \cup (8, \infty)$.

② On $(-\infty, 0)$, the mod sign is active, hence the inequality is

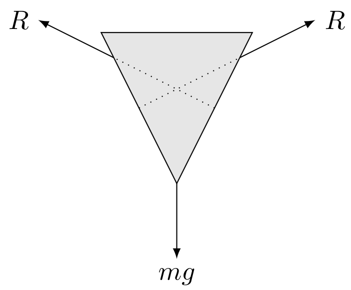
$$(5-x)(5+x) > 9.$$

For $x \in (-\infty, 0)$, this gives $x \in (-4, 0)$.

Combining these,

$$\begin{aligned} & \big(5-x\big)\big(5-|x|\big) > 9 \\ \Longrightarrow\ & x \in (-4, 2) \cup (8, \infty). \end{aligned}$$

4750. For $x > 0$, the curve is $y = 2\ln x$. Its gradient at $x = 1$ is $m = 2$. So, the sides of the cone have gradient 2. Since the funnel is smooth, it can only apply a normal reaction force to the plug. For $x > 0$, this is along a line of gradient $-\frac{1}{2}$. In cross-section, the force diagram for the plug is



Since the funnel is rotationally symmetrical, every other cross-section is identical. So, at all points around the circle of contact, the reaction force has the same 2 : 1 ratio of horizontal to vertical components. Hence, without loss of generality, we can work exclusively in the diagram above, with the two R forces together representing the total (not resultant, see note below) force applied, i.e. $2R = R_{\text{total}}$.

Resolving vertically,

$$\begin{aligned} & 2R \times \tfrac{1}{\sqrt{5}} - mg = 0 \\ \Longrightarrow\ & R_{\text{total}} = \sqrt{5}mg. \end{aligned}$$

By NIII, the total force exerted on the plug must be equal in magnitude to the total force exerted on the funnel. This force is spread over the circle of contact, which has a radius of 1 cm, and therefore a circumference of 2π cm. So, the force applied per centimetre of the circle of contact is

$$\frac{\sqrt{5}mg}{2\pi} \text{ N}.$$

— Nota Bene —

The *total* force is not the same as the *resultant* force. Because the plug pushes symmetrically in all horizontal directions, the resultant horizontal force is zero: the sum of the forces as vectors. The total horizontal force, however, as felt locally per centimetre of contact, is not zero: it depends on the *magnitudes* of the vectors.

An analogy: suppose you stood in a door-frame and pushed outwards on both sides, applying 10 N with each arm. You apply no *resultant* force to the door frame, because, adding vectors, $+10-10=0$ N. However, you do apply a *total* force to the door frame, because, adding scalars, $10+10=20$ N.

4751. (a) Every move adds or subtracts 1 from either the x or y coordinate. The parity of the sum $x+y$, therefore, changes by 1 each iteration. Initially, $x+y=0$, which is even. So, after five steps, $x+y$ must be odd. So, the probability of being at $(1,1)$ is zero.

(b) To end up at $(3,0)$, there must be at least three $\mathbf{i}$ steps. The remaining two can be $\pm\mathbf{i}$ or $\pm\mathbf{j}$. This gives two possible types of route to $(3,0)$:

① $\{\mathbf{i},\mathbf{i},\mathbf{i},\mathbf{i},-\mathbf{i}\}$. This outcome, in this order, has probability $1/4^5$. There are 5 orders of the set.

② $\{\mathbf{i},\mathbf{i},\mathbf{i},\mathbf{j},-\mathbf{j}\}$. This outcome, in this order, has probability $1/4^5$. Since three of the steps are identical, the number of orders of this set is $5!/3!$.

Hence, the total probability is

$$p=5\times\tfrac{1}{4}^5+\tfrac{5!}{3!}\times\tfrac{1}{4}^5=\tfrac{25}{1024}.$$

4752. Using various identities,

$$\begin{aligned}&4\tan 2\psi+3\cot\psi\sec^2\psi=0\\\implies&\frac{8\tan\psi}{1-\tan^2\phi}+\frac{3(1+\tan^2\psi)}{\tan\psi}=0\\\implies&8\tan^2\psi+3(1+\tan^2\psi)(1-\tan^2\phi)=0\\\implies&3\tan^4\psi-8\tan^2\psi+3=0.\end{aligned}$$

This is a quadratic in $\tan^2\psi$:

$$(3\tan^2\psi+1)(\tan^2\psi-3)=0.$$

The first factor has no real roots. The latter has

$$\begin{aligned}&\tan\psi=\pm\sqrt{3}\\\therefore\quad&\psi\in\left\{\tfrac{1}{3}\pi,\tfrac{2}{3}\pi,\tfrac{4}{3}\pi,\tfrac{5}{3}\pi\right\}.\end{aligned}$$

4753. Call the quartic $y=\text{g}(x)$. We know that $\text{g}''(x)$ is quadratic. It is zero at $x=p,q$, so must be a scalar multiple of ax^2+bx+c:

$$\begin{aligned}&\text{g}''(x)=kax^2+kbx+kc\\\implies&\text{g}'(x)=\tfrac{1}{3}kax^3+\tfrac{1}{2}kbx^2+kcx+d.\end{aligned}$$

We are told that $\text{g}'(0)=0$, so $d=0$. Integrating,

$$\text{g}(x)=\tfrac{1}{12}kax^4+\tfrac{1}{6}kbx^3+\tfrac{1}{2}kcx^2+e.$$

The curve $y=\text{g}(x)$ passes through the origin, so $e=0$. Also, the quartic is monic, so $\frac{1}{12}ka=1$, which gives $k=12/a$. The quartic is

$$y=x^4+\tfrac{2b}{a}x^3+\tfrac{6c}{a}x^2.$$

4754. (a) The derivative is

$$\text{f}'(x)=8x^3-6x^2+2x+1.$$

Setting this equal to 1,

$$\begin{aligned}&8x^3-6x^2+2x+1=1\\\implies&8x^3-6x^2+2x=0\\\implies&x(8x^2-6x+2)=0.\end{aligned}$$

This has a root $x=0$. There are no other real roots, as the quadratic has $\Delta=-28<0$. Hence, since $\text{f}'(x)$ is a positive cubic, $\text{f}'(x)>1$ for all $x>0$.

(b) The second derivative is

$$\text{f}''(x)=24x^2-12x+2.$$

This has $\Delta=-48<0$, so $\text{f}''(x)>0$ for all x. So, the gradient $\text{f}'(x)$ is increasing everywhere. We know, from part (a), that

$$x_n<x_{n+1}<x_{n+2}.$$

Since the gradient is increasing everywhere, the difference between subsequent terms of the sequence is increasing:

$$\begin{aligned}&x_{n+2}-x_{n+1}>x_{n+1}-x_n\\\implies&x_{n+2}+x_n>2x_{n+1},\text{ as required.}\end{aligned}$$

4755. Differentiating,

$$\dot{x}=(2k+1)t^{2k}-(2k-1)t^{2k-2}.$$

Solving for y intercepts

$$\begin{aligned}&t^{2k+1}-t^{2k-1}=0\\\implies&t^{2k-1}(t^2-1)=0\\\implies&t=0,\pm1.\end{aligned}$$

So, using the parametric integration formula, the area in question is given by

$$\begin{aligned}A&=2\int_0^1 t^{2k}\Big((2k+1)t^{2k}-(2k-1)t^{2k-2}\Big)\,dt\\&\equiv 2\int_0^1(2k+1)t^{4k}-(2k-1)t^{4k-2}\,dt\\&\equiv 2\left[\frac{2k+1}{4k+1}t^{4k+1}-\frac{2k-1}{4k-1}t^{4k-1}\right]_0^1\\&\equiv 2\left(\frac{2k+1}{4k+1}-\frac{2k-1}{4k-1}\right)\\&\equiv\frac{8k}{16k^2+1},\text{ as required.}\end{aligned}$$

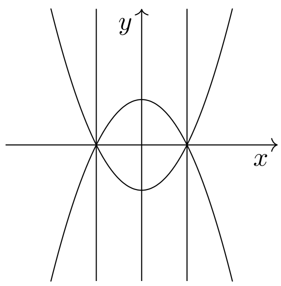
4756. The minimum possible value is zero, attained when A and B are mutually exclusive.

For the maximum value, events A and B coincide, and both are subsets of C. This gives a probability of $\frac{1}{2}$, which cannot be exceeded. All probabilities between these two extremes are attainable, as the overlap between A and B increases from zero to $\frac{1}{2}$. In set notation, $\mathbb{P}(A \cap B \cap C) \in [0, 1/2]$.

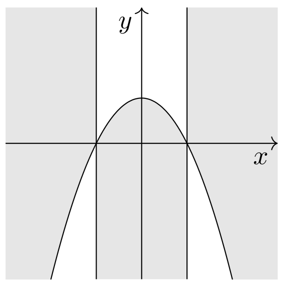
4757. For the fraction to be less than 1, we require one of the following, where N is the numerator and D the denominator:

- N positive, D negative,
- D positive, N negative,
- N and D both positive, with $N < D$,
- N and D both negative, with $N > D$.

The boundary equation is $y - x^2 + 1 = y + x^2 - 1$, which gives $x = \pm 1$. This is the boundary at $N = D$. There are also possible boundaries at $N = 0$ and $D = 0$. These are $y = x^2 - 1$ and $y = 1 - x^2$. So, the only locations at which there can be boundaries are



Testing the various regions, the solution set is



4758. Let p be the time of release. At this instant, the particle is at position $(3\sin kp, 19.6)$ with velocity $3k\cos kp$. In projectile motion, the vertical *suvat* is $-19.6 = -\frac{1}{2}gt^2$, so $t = 2$. The horizontal motion is at constant velocity. So, the landing position is

$$x = 3\sin kp + 6k\cos kp.$$

We need to choose k such that the range is $[-5, 5]$. So, the amplitude, in harmonic form, must be 5. This is the Pythagorean sum of the coefficients:

$$\begin{aligned} & 3^2 + (6k)^2 = 5^2 \\ \Longrightarrow\ & k = \pm\tfrac{2}{3}. \end{aligned}$$

4759. (a) The relevant identity is

$$\begin{aligned} & 3x^2 + 2xy + 3y^2 \\ \equiv\ & a(x+y)^2 + b(x-y)^2 \\ \equiv\ & (a+b)x^2 + (2a-2b)xy + (a+b)y^2. \end{aligned}$$

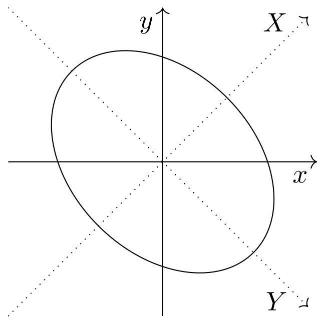
Equating coefficients, $a+b = 3$ and $2a-2b = 2$. This gives $a = 2$ and $b = 1$. So,

$$\text{f}(x,y) = 2(x+y)^2 + (x-y)^2.$$

(b) Let (X, Y) be new coordinate variables defined by $X = x + y$ and $Y = x - y$. The curve $\text{f}(x,y) = 1$ is then

$$2X^2 + Y^2 = 1.$$

This is an ellipse in the (X, Y) plane, whose axes are at 45° to those of the (x, y) plane:



4760. (a) For intersections,

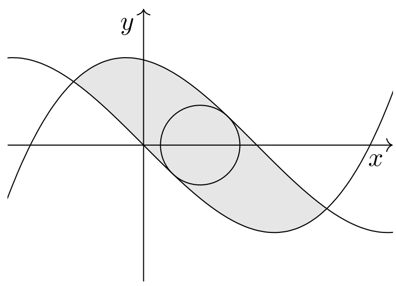
$$\begin{aligned} & x^3 - x = x^3 - 3kx^2 + 3k^2x - k^3 - x + k \\ \Longrightarrow\ & 3kx^2 - 3k^2x + k^3 - k = 0. \end{aligned}$$

Setting $\Delta = 0$,

$$\begin{aligned} & \left(-3k^2\right)^2 - 12k(k^3 - k) = 0 \\ \Longrightarrow\ & 12k^2 - 3k^4 = 0 \\ \Longrightarrow\ & k = 0, \pm 2. \end{aligned}$$

For $k = 0$, the curves coincide everywhere, so this value isn't relevant. The set of values of k for which the curves intersect is $[-2, 2]$.

(b) The second curve is a translation of the first by vector $\frac{1}{2}\mathbf{i}$. So, the problem has rotational symmetry around the point $C : (1/4, 0)$. Hence, this must be the centre of the largest circle.



The radius is greatest if the circle is tangent to the curve. So, we need to find the normal to the curve $y = x^3 - x$ which passes through C. The normal at $x = a$ is

$$y - a^3 + a = -\frac{1}{3a^2 - 1}(x - a).$$

Substituting point C,

$$\begin{aligned} &-a^3 + a = -\frac{1}{3a^2-1}\left(\tfrac{1}{4} - a\right) \\ \Longrightarrow\ & (-a^3 + a)(3a^2 - 1) = a - \tfrac{1}{4} \\ \Longrightarrow\ & 3a^5 - 4a^3 + 2a - \tfrac{1}{4} = 0. \end{aligned}$$

This is not analytically solvable. N-R is

$$a_{n+1} = a_n - \frac{3a_n^5 - 4a_n^3 + 2a_n - \frac{1}{4}}{15a_n^4 - 12a_n^2 + 2}.$$

Running this with $a_0 = 0$, we get $a_1 = 0.125$ and then $a_n \to 0.129266$. This gives the point of tangency with $y = x^3 - x$ as

$$(0.129266, -0.127106).$$

The distance of this point from $(1/4, 0)$ is

$$r = 0.1753 \text{ (4sf)}.$$

4761. The perimeter is

$$\begin{aligned} 2s &= a + b + c \\ &= n(m^2 + k^2) + m(n^2 + k^2) \\ &\qquad + (m+n)(mn - k^2) \\ &\equiv nm^2 + nk^2 + mn^2 + mk^2 \\ &\qquad + m^2n + mn^2 - mk^2 - nk^2 \\ &\equiv 2mn^2 + 2m^2n. \end{aligned}$$

So, the semiperimeter is $s = mn(m + n)$. Hence, the factors in the radicand of Heron's formula are

$$\begin{aligned} (s-a) &= mn(m+n) - n(m^2 + k^2) \\ &\equiv n(mn - k^2), \\ (s-b) &= mn(m+n) - m(n^2 + k^2) \\ &\equiv m(mn - k^2), \\ (s-c) &= mn(m+n) - (m+n)(mn - k^2) \\ &\equiv (m+n)k^2. \end{aligned}$$

Therefore, the radicand is

$$\begin{aligned} &s(s-a)(s-b)(s-c) \\ =\ & mn(m+n)n(mn-k^2)m(mn-k^2)(m+n)k^2 \\ \equiv\ & m^2n^2k^2(m+n)^2(mn-k^2)^2. \end{aligned}$$

Hence, by Heron's formula,

$$A = mnk(m+n)(mn - k^2), \text{ as required.}$$

4762. The boundary equation may be factorised as

$$xy(x^2 - y^2) = 1.$$

Squaring this,

$$x^2y^2(x^2 - y^2)^2 = 1.$$

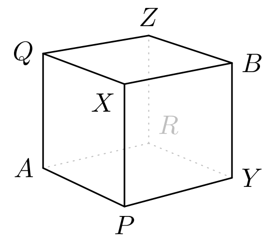
Substituting the equation of circle,

$$\begin{aligned} & x^2(2 - x^2)(2x^2 - 2)^2 = 1 \\ \Longrightarrow\ & 4x^8 - 16x^6 + 20x^4 - 8x^2 + 1 = 0 \\ \Longrightarrow\ & (2x^4 - 4x^2 + 1)^2 = 0. \end{aligned}$$

The biquadratic $2x^4 - 4x^2 + 1$ has $\Delta = 8 > 0$, so $x^2 = 1 \pm \sqrt{2}/2$ satisfies the above. Both of these values $1 \pm \sqrt{2}/2$ are positive, so the biquadratic has four distinct real roots.

In the octic, each of these is a double root, giving a point of tangency. Hence, the region $x^3y - xy^3 \geq 1$ is tangent to $x^2 + y^2 = 2$ at four points.

4763. There are three vertices which share an edge with A. Call these P, Q, R. Then call the remaining vertices X, Y, Z.



From A, there are three symmetrical choices to P, Q, R. Choose P. From P, there are two choices, X and Y. Again, these are symmetrical. Choose X. At this point, there are three options:

① $A \to P \to X \to B$
② $A \to P \to X \to Q \to Z \to B$
③ $A \to P \to X \to Q \to Z \to R \to Y \to B$.

There were six symmetrical choices at the start, which gives $6 \times 3 = 18$ routes in total.

4764. For $y = k$ to intersect the curve three times, the following equation must have three distinct roots:

$$\begin{aligned} & \frac{1}{x} - \frac{1}{x^3} - k = 0 \\ \Longrightarrow\ & kx^3 - x^2 + 1 = 0. \end{aligned}$$

For a cubic to have three distinct roots, it must have stationary values which are +ve and −ve. Setting the derivative to zero,

$$\begin{aligned} & 3kx^2 - 2x = 0 \\ \Longrightarrow\ & x(3kx - 2) = 0 \\ \Longrightarrow\ & x = 0, \tfrac{2}{3k}. \end{aligned}$$

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Evaluating the cubic,

$$kx^3 - x^2 + 1\big|_{x=0} = 1 > 0$$
$$kx^3 - x^2 + 1\big|_{x=2/3k} = -\tfrac{4}{27k^2} + 1.$$

So, for three distinct roots,

$$-\tfrac{4}{27k^2} + 1 > 0$$
$$\iff |k| < \tfrac{2}{3\sqrt{3}}.$$

We exclude $k = 0$, for which the above argument is not well defined; the curve has no x intercepts. The full solution, therefore, is

$$k \in \left(-\tfrac{2}{3\sqrt{3}}, 0\right) \cup \left(0, \tfrac{2}{3\sqrt{3}}\right).$$

4765. (a) Squaring both sides,

$$\frac{3-\sqrt{5}}{8} = a^2 + 5b^2 + 2ab\sqrt{5}$$
$$\implies 3 - \sqrt{5} = 8a^2 + 40b^2 + 16ab\sqrt{5}.$$

Equating rational and irrational coefficients,

$$3 = 8a^2 + 40b^2,$$
$$-1 = 16ab.$$

Solving simultaneously, the relevant values are

$$a = b = \tfrac{1}{4}.$$

(b) We use the identity $\cos 2\theta \equiv 1 - 2\sin^2\theta$, with $\theta = 18°$. This gives

$$1 - 2\sin^2 18° = \tfrac{1}{4}\big(1 + \sqrt{5}\big)$$
$$\implies 4 - 8\sin^2 18° = 1 + \sqrt{5}$$
$$\implies \sin^2 18° = \frac{3-\sqrt{5}}{8}.$$

Taking the positive root,

$$\sin 18° = \frac{\sqrt{3-\sqrt{5}}}{2\sqrt{2}}.$$

The result from part (a) converts this to

$$\sin 18° = \tfrac{1}{4}(\sqrt{5} - 1), \text{ as required.}$$

4766. The relevant integral is

$$A = 6\int_0^a \frac{1}{1+x^2}\,dx.$$

Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. The new limits are $\theta = 0$ to $\theta = \arctan a$. The area is

$$6\int_0^{\arctan a} \frac{1}{1+\tan^2\theta}\sec^2\theta\,d\theta$$
$$= 6\int_0^{\arctan a} 1\,d\theta$$
$$= 6\arctan a.$$

The value of the integral is given as 2π, so

$$6\arctan a = 2\pi$$

Hence, $a = \tan \pi/3$, which is $\sqrt{3}$.

4767. The equation $\mathrm{h}''(x) = 0$ has exactly two roots. Call them a and b. Assume, for a contradiction, that both are roots of odd multiplicities m and n. Taking out factors of $(x-a)^m$ and $(x-b)^n$, the equation $\mathrm{h}''(x) = 0$ can be expressed as

$$(x-a)^m(x-b)^n\,\mathrm{p}(x) = 0,$$

where $\mathrm{p}(x)$ has no factors of $(x-a)$ or $(x-b)$. But, since $m+n$ is even, $\mathrm{p}(x)$ is a polynomial of odd degree. So, it must have a root. This root is neither a nor b, nor can it be anything else. This is a contradiction.

So, one root of $\mathrm{h}''(x)$ must have even multiplicity. At this root, $\mathrm{h}''(x)$ does not change sign. Hence, $y = \mathrm{h}(x)$ has exactly one point of inflection. □

4768. Assume, for a contradiction, that the cube root of 5 is rational, and can be written as p/q, where p and q are integers. This gives

$$5q^3 = p^3.$$

Consider the number of factors of 5 on each side of this equation. The number of factors of 5 in a cube must be a multiple of 3. So, equating the number of factors of 5, we get $3m + 1 = 3n$, for some integers m and n. This is a contradiction: the RHS is a multiple of 3, but the LHS isn't. So, the cube root of 5 must be irrational. □

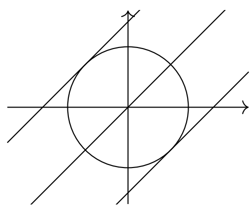
4769. The second equation is a circle of radius 2 centred at the origin. Factorising the first equation,

$$(x-y)\big((x-y)^2 - k\big) = 0$$
$$\iff x - y = 0, \pm\sqrt{k}.$$

So, the locus of the first equation is a set of lines:

$$y = x,$$
$$y = x \pm \sqrt{k}.$$

The former always intersects twice with the circle. To get two more points of intersection, we require $k > 0$. In this case, the two lines $y = x \pm \sqrt{k}$ are symmetrical. So, each must generate exactly one point of intersection with the circle. They must, therefore, be tangential to it:



This occurs at $k = 8$.

4770. Since b is a root, we know that

$$\begin{aligned} ab^3 + ab^2 + ab + b &= 0 \\ \therefore\quad ab^2 + ab + a + 1 &= 0. \end{aligned}$$

This is a quadratic in b. It must have at least one root. Setting $\Delta > 0$,

$$\begin{aligned} & a^2 - 4a(a+1) \geq 0 \\ \Longrightarrow\ & -a(3a+4) \geq 0 \\ \Longrightarrow\ & -\tfrac{4}{3} \leq a \leq 0, \text{ as required.} \end{aligned}$$

4771. Using the cosine rule for $\triangle ABC$,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

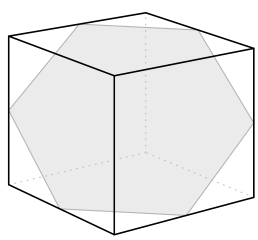
And for $\triangle ABM$,

$$\begin{aligned} m^2 &= \tfrac{1}{4}a^2 + c^2 - ac\cos B \\ &= \tfrac{1}{4}a^2 + c^2 - \frac{a^2 + c^2 - b^2}{2} \\ &\equiv -\tfrac{1}{4}a^2 + \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2. \end{aligned}$$

Taking the positive square root,

$$m = \sqrt{\frac{b^2 + c^2}{2} - \frac{a^2}{4}}, \text{ as required.}$$

4772. The symmetry of the cube and such a regular hexagon dictates that the vertices of the hexagon must be at the midpoints of edges of the cube.



If the cube has side length 1, then the hexagon has side length $\sqrt{2}/2$. Simplifying, the ratio is $\sqrt{2} : 1$.

4773. (a) The indefinite integral of g is G. So,

$$\begin{aligned} \int_0^a \mathrm{g}(x)\,dx &= \Big[\mathrm{G}(x)\Big]_0^a \\ &= \mathrm{G}(a) - \mathrm{G}(0) \\ &= b. \end{aligned}$$

(b) Integrating by inspection,

$$\begin{aligned} \int_0^a x\,\mathrm{g}(x^2)\,dx &= \Big[\tfrac{1}{2}\,\mathrm{G}\left(x^2\right)\Big]_0^a \\ &\equiv \tfrac{1}{2}\,\mathrm{G}\left(a^2\right) - \tfrac{1}{2}\,\mathrm{G}(0) \\ &= \tfrac{1}{2}b^2. \end{aligned}$$

— Nota Bene —

This result could also be attained, and further understood, by using the substitution $u = x^2$.

4774. We address the factors one by one:

① $(x^2 + x + k^2 + 1)$ is a quadratic with

$$\begin{aligned} \Delta &= 1 - 4(k^2 + 1) \\ &= -4k^2 - 3 < 0. \end{aligned}$$

So, this factor has no real roots.

② $(x^3 - x - k^2 - 1)$ is cubic, so it must have at least one real root. For stationary values,

$$\begin{aligned} & 3x^2 - 1 = 0 \\ \Longrightarrow\ & x = \pm\tfrac{1}{\sqrt{3}}. \end{aligned}$$

Substituting in, the stationary values are

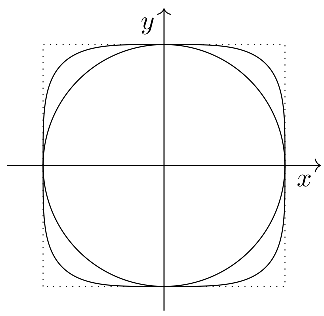
$$\mp\tfrac{2}{3\sqrt{3}} - k^2 - 1.$$

Since $2/3\sqrt{3} < 1$, both of the stationary values are negative, whatever the value of k. Hence, this factor has exactly one root.

③ Since $k^4 > 0$ and $x^6 \geq 0$, this factor cannot have any real roots.

The total number of roots, therefore, is 1.

4775. The equations $x^{2n} + y^{2n} = 1$ are akin to circles. The first, $n = 1$, is the unit circle centred on the origin. The second is $x^4 + y^4 = 1$, which is outside or on the unit circle:



The pattern continues, with the curves for higher powers outside the curves for lower powers. In the limit of large n, the curves tend towards the dotted square above. Hence, $\lim\limits_{n\to\infty} A_n = 4$.

— Nota Bene —

For a more detailed proof, consider the point of intersection of $x^{2n} + y^{2n} = 1$ with $y = x$, which is the point on the curve furthest from the origin. Solving simultaneously gives $2x^{2n} = 1$, so

$$x = \sqrt[2n]{\tfrac{1}{2}}.$$

As $x \to \infty$, this tends to 1. So, the relevant point of intersection tends towards the point $(1, 1)$. The same applies in all four quadrants, giving explicit proof of the result above.

4776. (a) Squaring the y equation,

$$\begin{aligned} y^2 &= \sin^2 2t \\ \Longrightarrow y^2 &= 4\sin^2 t\cos^2 t \\ \Longrightarrow y^2 &= 4x^2\big(1-x^2\big). \end{aligned}$$

(b) The lobe with $x \geq 0$ has t limits $t = 0$ to $t = \pi$. The x derivative is $\dot{x} = -\sin t$. So, the area of both lobes is given by

$$\begin{aligned} A &= 2\int_0^\pi \sin 2t \cdot -\sin t \, dt \\ &= -4\int_0^\pi \sin^2 t \cot t \, dt. \end{aligned}$$

Integrating by inspection,

$$\begin{aligned} A &= -4\Big[\tfrac{1}{3}\cos^3 t\Big]_0^\pi \\ &= -4\big(-\tfrac{1}{3}-\tfrac{1}{3}\big) \\ &= \tfrac{8}{3}, \text{ as required.} \end{aligned}$$

4777. Consider the domain $x \in (0, 1]$. Over this domain, the range of $\ln x$ is $(\infty, 0]$. Using this range as the domain of the sine function and the fact that the ln function is one-to-one, the value of $\sin(\ln x)$ must oscillate continuously between -1 and 1 infinitely many times in between $x = 0$ and $x = 1$.

The curve $y = \sin(\ln x)$ must therefore cross the line $y = x$ infinitely many times between $x = 0$ and $x = 1$. Each of these gives a distinct root. □

4778. From the information given, the functions are

$$\begin{aligned} \mathrm{f}(x) &= (x-a)(x-b), \\ \mathrm{g}(x) &= (x-b)(x-c). \end{aligned}$$

The equation $\mathrm{f}(x) + \mathrm{g}(x) = 0$ is

$$\begin{aligned} &(x-a)(x-b) + (x-b)(x-c) = 0 \\ \Longrightarrow\ &(x-b)\big((x-a)+(x-c)\big) = 0. \end{aligned}$$

This has a root at $x = b$. And the second factor also has a root, at $2x - a - c = 0$. This gives $x = \frac{1}{2}(a + c)$. We need to show that these roots are distinct, i.e. that

$$b \neq \frac{a+c}{2}.$$

Assume, for a contradiction, that $2b = a+c$. Using the fact of GP to substitute $b = ar$ and $c = ar^2$,

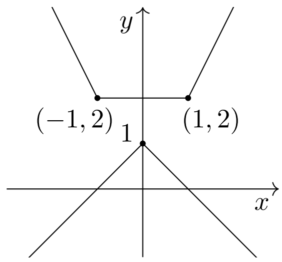
$$\begin{aligned} &2ar = a + ar^2 \\ \Longrightarrow\ &a(r^2 - 2r + 1) = 0 \\ \Longrightarrow\ &a(r-1)^2 = 0 \\ \Longrightarrow\ &a = 0 \text{ or } r = 1. \end{aligned}$$

We are told that neither is the case, so this is a contradiction. Hence, the roots provided by the two factors are distinct, giving exactly two roots overall. □

4779. We sketch the LHS and RHS. Consider

$$y = |x-1| + |x+1|.$$

This has changes of behaviour at $x = \pm 1$, at which points its y value is 2. Outside the domain $[-1, 1]$, it has gradient ± 2. Inside, the gradient is 0. The graphs of both sides are



As can be seen, there are no x values for which the LHS and RHS of the equation are equal.

4780. Eliminating t from horizontal and vertical *suvat*s, the equation of the trajectory is

$$y = x\tan\theta - \frac{gx^2}{2u^2}(\tan^2\theta + 1).$$

Since (p, q) lies on this trajectory,

$$q = p\tan\theta - \frac{gp^2}{2u^2}(\tan^2\theta + 1).$$

Writing this as a quadratic in $\tan\theta$,

$$gp^2\tan^2\theta - 2pu^2\tan\theta + 2qu^2 + gp^2 = 0.$$

The quadratic formula gives

$$\begin{aligned} \tan\theta &= \frac{2pu^2 \pm \sqrt{4p^2u^2 - 4gp^2(2qu^2 + gp^2)}}{2gp^2} \\ &\equiv \frac{u^2 \pm \sqrt{u^2 - g(2qu^2 + gp^2)}}{gp}, \text{ as required.} \end{aligned}$$

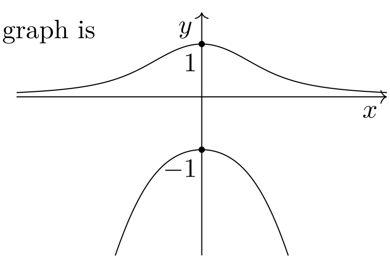
4781. Solving as a quadratic in y,

$$\begin{aligned} &y^2 + x^2y - 1 = 0 \\ \Longrightarrow\ &y = \frac{-x^2 \pm \sqrt{x^4+4}}{2}. \end{aligned}$$

This gives two distinct curves, each of which is symmetrical in the y axis. Their y intercepts are ± 1. In each, as $x \to \pm\infty$, the $+4$ under the square root becomes negligible, and $\sqrt{x^4+4} \to x^2$ from above. Hence, as $x \to \pm\infty$, the behaviour is:

- with the positive root, $y \to 0^+$,
- with the negative root, $y \to -x^2$.

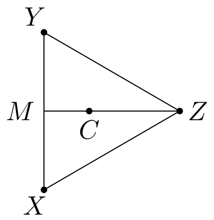
So, the graph is



4782. Let A be the apex, C be the centre of the base, X, Y, Z be the vertices of the base, and M be the midpoint of XY. The gradient triangles are

$\triangle ACM$ for the sloped faces,
$\triangle ACZ$ for the sloped edges.

Since both triangles have the same vertical height AC, this doesn't enter into calculations. The only relevant consideration is the ratio $CM : CZ$.



The centroid divides medians in the ratio $1 : 2$. So, the gradient of a face is twice that of an edge. □

4783. Using a double-angle formula, we can factorise the LHS of the equation of the curve:

$$\begin{aligned} &\sin 2x - 2e^y \sin x + e^{-y}\cos x - 1 = 0 \\ \iff\ & 2\sin x\cos x - 2e^y \sin x + e^{-y}\cos x - 1 = 0 \\ \iff\ & \big(\cos x - e^y\big)\big(2\sin x + e^{-y}\big) = 0. \end{aligned}$$

So, the two branches of the curve are

$$\begin{aligned} e^y &= \cos x, \\ e^{-y} &= -2\sin x. \end{aligned}$$

The point of self-tangency is on both, so we solve simultaneously. Multiplying the equations,

$$\begin{aligned} & 1 = -2\sin x \cos x \\ \implies\ & \sin 2x = -1. \end{aligned}$$

Taking the relevant value, $x = -\frac{\pi}{4}$. Substituting this back into either curve gives coordinates

$$A : \Big(-\pi/4, \ln(\sqrt{2}/2)\Big).$$

4784. We use the product

$$\big(x-y\big)\big(x^3 + x^2y + xy^2 + y^3\big) \equiv x^4 - y^4.$$

The first-principles limit is

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^{\frac14} - x^{\frac14}}{h}.$$

Multiplying top and bottom by the same factor, the expression inside the limit is

$$\begin{aligned} &\frac{(x+h) - x}{h\Big((x+h)^{\frac34} + (x+h)^{\frac24}x^{\frac14} + (x+h)^{\frac14}x^{\frac24} + x^{\frac34}\Big)} \\ \equiv\ & \frac{1}{(x+h)^{\frac34} + (x+h)^{\frac24}x^{\frac14} + (x+h)^{\frac14}x^{\frac24} + x^{\frac34}}. \end{aligned}$$

We can now take the limit, giving

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x^{\frac34} + x^{\frac34} + x^{\frac34} + x^{\frac34}} \\ &\equiv \tfrac14 x^{-\frac34}. \end{aligned}$$

4785. The derivatives are

$$\begin{aligned} x &= e^{-t}\sin 2t \\ \dot{x} &= e^{-t}(2\cos 2t - \sin 2t) \\ \ddot{x} &= e^{-t}(-4\cos 2t - 3\sin 2t). \end{aligned}$$

Substituting into the LHS of the DE, we multiply through by e^t. This leaves

$$\begin{aligned} (-4\cos 2t - 3\sin 2t) + a(2\cos 2t - \sin 2t) \\ + b\sin 2t \equiv 0. \end{aligned}$$

Equating coefficients of $\cos 2t$ and $\sin 2t$,

$$\begin{aligned} \cos 2t &: -4 + 2a = 0, \\ \sin 2t &: -3 - a + b = 0. \end{aligned}$$

Solving these, $a = 2$ and $b = 5$.

— Nota Bene —

The use of the identity symbol in the above is key. A DE must hold for *all* values of the independent variable, which allows us to equate coefficients. There's quite a subtle point here. In a DE, the equality, which is written (correctly) as $=$ rather than $\equiv$, has a meaning different from the equals sign in $t^2 = 4$ and also different from the identity symbol in $t^2 - 4 \equiv (t+2)(t-2)$. It has both meanings, depending on how you look at it:

① identity in the variable t,
② equation in the function $t \mapsto x$.

In a DE such as $\frac{dy}{dx} = 2x$, we require the thing to be an identity in x, but are nevertheless asking a question about the functional relationship between y and x, which might or might not be true. In other words, $y = x^2 + c$ makes the two sides of the DE identical, but $y = x^3 + c$ would not make them identical. So, we use $=$ in the initial DE, because it does not always hold:

$$\frac{dy}{dx} = 2x.$$

But, having substituted a potential relationship in, e.g. $y = x^k + c$, we require an identity:

$$kx^{k-1} \equiv 2x.$$

4786. Firstly, we need to find a root. Calling the given expression $\mathrm{f}(x)$, the N-R iteration is

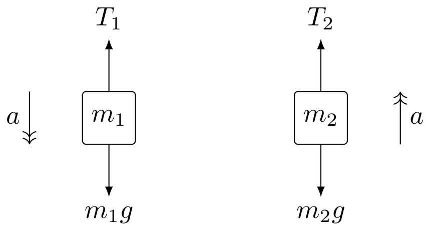
$$x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}.$$

Running the iteration with almost any starting point yields $x_n \to 6$. Taking out a factor of $(x-6)$,

$$\begin{aligned} & x^5 - 6x^4 + 6x^3 - 36x^2 + 9x - 54 \\ \equiv\ & (x-6)(x^4 + 6x^2 + 9) \\ \equiv\ & (x-6)(x^2+3)^2. \end{aligned}$$

The squared quadratic factor is irreducible, so we have factorised fully.

4787. The force diagrams are



The upward force exerted by the pulley on the string is $T_1 + T_2$. So, this is the downwards force exerted by the string on the pulley. If the system moves, then friction is at maximal. The tension T_1 is necessarily larger than the tension T_2, and the difference, we are told, is

$$T_1 - T_2 = \mu(T_1 + T_2).$$

The equation of motion for the first block is

$$\begin{aligned} m_1g - T_1 &= m_1a \\ \implies T_1 &= m_1g - m_1a. \end{aligned}$$

And for the second block:

$$\begin{aligned} T_2 - m_2g &= m_2a \\ \implies T_2 &= m_2g + m_2a. \end{aligned}$$

Substituting these into the friction equation,

$$\begin{aligned} & m_1g - m_1a - (m_2g + m_2a) \\ &\quad = \mu(m_1g - m_1a + m_2g + m_2a) \\ \implies & am_1(1-\mu) + am_2(1+\mu) \\ &\quad = \big(m_1(1-\mu) - m_2(1+\mu)\big)g \\ \implies & a = \frac{m_1(1-\mu) - m_2(1+\mu)}{m_1(1-\mu) + m_2(1+\mu)}g. \end{aligned}$$

Consider a smooth system with masses $M < m$. Taking NII along the string,

$$a = \frac{M - m}{M + m}g.$$

Substituting $M = m_1(1-\mu)$ and $m = m_2(1+\mu)$, we get the acceleration of the rough system. So, if it moves, the rough system with masses m_1 and m_2 behaves like a smooth system with masses $m_1(1-\mu)$ and $m_2(1+\mu)$. □

4788. Multiplying by x^3, this is a cubic in x^9:

$$\begin{aligned} & x^{27} - 3x^{18} + 3x^9 - 1 = 0 \\ \implies & (x^9 - 1)^3 = 0 \\ \implies & x^9 = 1 \\ \implies & x = 1. \end{aligned}$$

4789. Using compound-angle formulae, the first term is

$$\begin{aligned} & \sin^2(x+y) \\ \equiv\ & (\sin x \cos y + \cos x \sin y)^2 \\ \equiv\ & \sin^2 x \cos^2 y + 2 \sin x \cos x \sin y \cos y \\ & \qquad + \cos^2 x \sin^2 y. \end{aligned}$$

The second term is

$$\begin{aligned} & \cos^2(x-y) \\ \equiv\ & (\cos x \cos y + \sin x \sin y)^2 \\ \equiv\ & \cos^2 x \cos^2 y + 2 \sin x \cos x \sin y \cos y \\ & \qquad + \sin^2 x \sin^2 y. \end{aligned}$$

Adding the above, the first and third terms of each expansion combine, via the first Pythagorean trig identity, to give $\sin^2 y + \cos^2 y$, and then to give 1. So, the original equation is

$$\begin{aligned} & 4 \sin x \cos x \sin y \cos y = 0 \\ \implies & \sin 2x \sin 2y = 0 \\ \implies & \sin 2x = 0 \text{ or } \sin 2y = 0. \end{aligned}$$

This implies that one of x or y is a multiple of $\frac{\pi}{2}$.

4790. Since the quartic has two local minima on the x axis, it has two double roots. So, we must be able to write

$$\begin{aligned} & 36x^4 + 12x^3 - 11x^2 - 2x + 1 \\ \equiv\ & 36(x-a)^2(x-b)^2. \end{aligned}$$

Equating coefficients,

$$\begin{aligned} & x^3 : 12 = 36(-2a - 2b) \implies a + b = -\tfrac{1}{6}, \\ & x^0 : 1 = 36a^2b^2. \end{aligned}$$

Solving these, a and b are $1/3$ and $-1/2$. So, these are the x coordinates of the local minima.

4791. Let $u = x + y$. Differentiating with respect to x,

$$\begin{aligned} \frac{du}{dx} &= 1 + \frac{dy}{dx} \\ \implies \frac{dy}{dx} &= \frac{du}{dx} - 1. \end{aligned}$$

Enacting the substitution,

$$\begin{aligned} & \frac{du}{dx} - 1 = u \\ \implies & \frac{du}{dx} = u + 1 \\ \implies & \int \frac{1}{u+1}\, du = \int 1\, dx \\ \implies & \ln|u+1| = x + c \\ \therefore\ & u + 1 = Ae^x. \end{aligned}$$

This gives the general solution as

$$\begin{aligned} x + y = 1 &= Ae^x \\ \implies y &= Ae^x - x - 1. \end{aligned}$$

4792. Adding the equations,

$$x + y = 4t$$
$$\implies t = \tfrac{1}{4}(x + y).$$

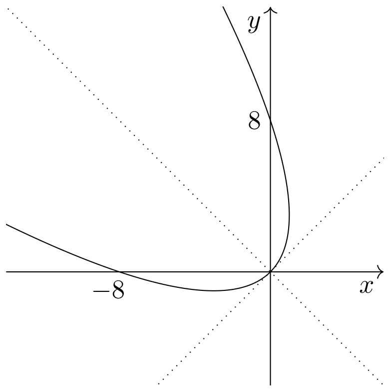
Subtracting the equations,

$$y - x = 2t^2.$$

Substituting the former into the latter,

$$y - x = \tfrac{1}{8}(x + y)^2.$$

The $x+y$ and $y-x$ axes are perpendicular, at 45° to x and y. So, the parametric equations define a parabola at 45° to the x and y axes:



4793. We can rewrite the fraction as

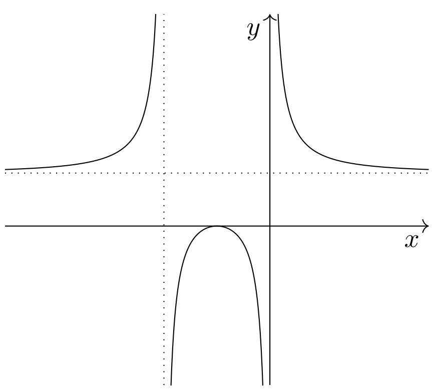
$$\begin{aligned} \mathrm{f}(x) &= \frac{x^2 + 2x + 1}{x^2 + 2x} \\ &\equiv 1 + \frac{1}{x(x+2)}. \end{aligned}$$

The range of $x(x + 2)$ is $[-1, \infty)$. So, the range of its reciprocal is $(-\infty, -1] \cup (0, \infty)$. Adding 1 to this, the range of the function f is

$$(-\infty, 0] \cup (1, \infty).$$

— Nota Bene —

The graph $y = \mathrm{f}(x)$ is



The asymptotes are $x = 0$, $x = -2$ and $y = 1$.

4794. (a) The boundary gradients are $m = 1$ and $m = 2$. Using a calculator, for $M \sim N(0, 1)$,

$$\mathbb{P}(1 < M < 2) = 0.136 \text{ (3sf)}.$$

(b) Using the conditional probability formula,

$$\begin{aligned} &\mathbb{P}(\text{steeper than } y = x \mid \text{+ve gradient}) \\ &= \frac{\mathbb{P}(M > 1)}{\mathbb{P}(M > 0)} \\ &= \frac{0.158655}{0.5} \\ &= 0.317 \text{ (3sf)}. \end{aligned}$$

4795. Let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta\, d\theta$. Enacting the substitution,

$$\begin{aligned} &\int \frac{1}{x\sqrt{x^2 - 1}}\, dx \\ &= \int \frac{1}{\sec\theta\sqrt{\sec^2\theta - 1}} \sec\theta\tan\theta\, d\theta. \end{aligned}$$

Using the second Pythagorean identity, this is

$$\begin{aligned} &\int \frac{1}{\sqrt{\tan^2\theta}} \tan\theta\, d\theta \\ &= \int 1\, d\theta \\ &= \theta + c \\ &= \operatorname{arcsec} x + c. \end{aligned}$$

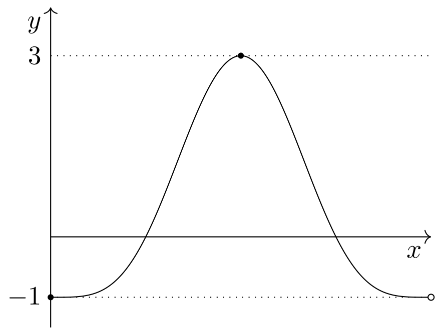
4796. The factorisation is

$$\begin{aligned} &-x^3y - xy^2 - xy + y + 1 + x^2 \\ &\equiv (x^2 + y + 1)(1 - xy). \end{aligned}$$

4797. Consider the curve $y = \cos^2 x - 2\cos x$. For SPs,

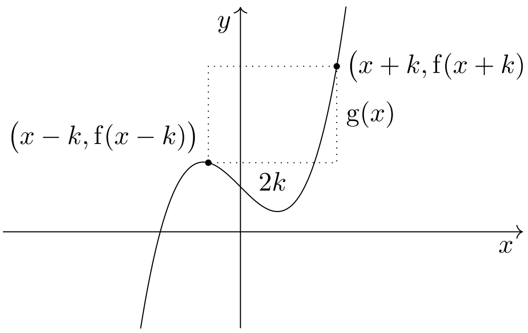
$$\begin{aligned} &-2\cos x \sin x + 2\sin x = 0 \\ \implies &\sin x(1 - \cos x) = 0 \\ \implies &\sin x = 0 \text{ or } \cos x = 1. \end{aligned}$$

In $[0, 2\pi)$, this gives $x = 0, \pi$. So, the stationary values are -1 and 3. The curve $y = \cos^2 x - 2\cos x$ looks as follows:



For the equation $\cos^2 x - 2\cos x + k = 0$ to have exactly one root in $[0, 2\pi)$, the line $y = -k$ must intersect this curve exactly once over this domain. So, $-k = -1, 3$. This gives $k \in \{-3, 1\}$.

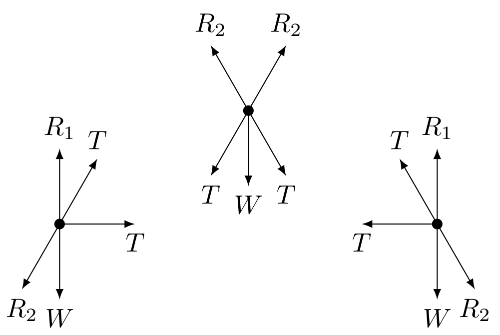
4798. Consider a graph of $y = \mathrm{f}(x)$. The values $x + k$ and $x - k$ are $2k$ apart. The quantity $\mathrm{g}(x)$ is then the y distance between such a pair of points:



For any polynomial function f and constant k, the leading coefficients of $\mathrm{f}(x + k)$ and $\mathrm{f}(x - k)$ are the same. Hence, the difference between them is a quadratic function. From the graph, we can see that this is a positive quadratic: the difference $\mathrm{g}(x)$ must increase as the gradient of the curve increases, which happens as $x \to \pm\infty$.

By symmetry, the minimum of the quadratic $\mathrm{g}(x)$ must occur at the point of inflection of $\mathrm{f}(x)$. Again by symmetry, this is halfway between the SPs. Therefore, $\mathrm{g}(x)$ is minimised at $\frac{1}{2}(p + q)$. □

4799. If the tension is too low, then equilibrium will be broken: the upper core will sink downwards, thereby separating the two lower cores. If the tension is high, then the two lower cores will be pressed against one another. In the boundary case between these regimes, the cores are as depicted, but the reaction force between the two lower cores is zero. The force diagrams are



Resolving horizontally for one of the lower cores,

$$\begin{aligned} &T + \tfrac{1}{2}T - \tfrac{1}{2}R_2 = 0 \\ \Longrightarrow\ &R_2 = 3T. \end{aligned}$$

Resolving vertically for the upper core,

$$\sqrt{3}R_2 - \sqrt{3}T - W = 0.$$

Substituting $R_2 = 3T$,

$$\begin{aligned} &3\sqrt{3}T - \sqrt{3}T - W = 0 \\ \Longrightarrow\ &2\sqrt{3}T = W \\ \Longrightarrow\ &T = \tfrac{\sqrt{3}}{6}W, \text{ as required.} \end{aligned}$$

4800. Let $u = b + e^{kx}$, giving $du = ke^{kx}\,dx$. We can rewrite this as

$$dx = \frac{du}{k(u - b)}.$$

Enacting the substitution,

$$\begin{aligned} &\int \frac{a + e^{kx}}{b + e^{kx}}\,dx \\ = &\int \frac{a + u - b}{ku(u - b)}\,du. \end{aligned}$$

Writing in partial fractions,

$$\begin{aligned} &\frac{a + u - b}{ku(u - b)} \equiv \frac{A}{u} + \frac{B}{u - b} \\ \Longrightarrow\ &a + u - b \equiv Ak(u - b) + Bku. \end{aligned}$$

Equating coefficients of u, $1 = Ak + Bk$. Equating the constant terms, $a - b = -Abk$. The latter gives

$$A = \frac{b - a}{bk}.$$

The former then gives

$$B = \frac{1}{k} - A = \frac{b}{bk} - \frac{b - a}{bk} = \frac{a}{bk}.$$

So, the integral is

$$\begin{aligned} &\int \frac{b - a}{bku} + \frac{a}{bk(u - b)}\,du \\ = &\frac{b - a}{bk}\ln|u| + \frac{a}{bk}\ln|u - b| + c \\ = &\frac{b - a}{bk}\ln|b + e^{kx}| + \frac{a}{bk}\ln|e^{kx}| + c \\ \equiv &\frac{(b - a)\ln|b + e^{kx}| + akx}{bk} + c, \text{ as required.} \end{aligned}$$

——— End of 48th Hundred ———